Lecture 12

- Trade: Option Spread, Vertical Spread or Straddle
- Portfolio Mgmt. with Options (RSD, p.483-5)
- The Greeks: Delta, Gamma, Theta etc. (RSD, p.485-97)
- Greeks for Option Spreads (RSD, p.499-511)

Portfolio Management with Options

One of the most useful applications of the Black-Scholes formula involves using the partial derivatives of the formula (the Greeks) to analyze and design portfolios/positions containing derivative securities.

For portfolios containing an arbitrary number of securities, define the value function *V* where V_i is the value of one unit of security *i* and n_i the number of units held where *(n* < 0) is a short position*:*

$$V = n_1 V_1 + n_2 V_2 + n_3 V_3 + \ldots\ldots$$

The Greeks for a Portfolio

- By taking partial derivatives of the value function *V* it is possible to specify the Greeks for the portfolio.
  - An example of this has already been used in defining the riskless hedge portfolio.
  - More generally, in the **two security case**, the delta of the portfolio can be derived:

$$\Delta_V = \frac{\partial V}{\partial S} = n_1 \frac{\partial V_1}{\partial S} + n_2 \frac{\partial V_2}{\partial S} = n_1 \Delta_1 + n_2 \Delta_2$$

The Greeks for Individual Call Options

- Delta, theta and gamma are names used to refer to the most commonly referenced partial derivatives.
- The partial derivatives are referred to as **Greeks** after the symbols used identify the derivatives.
- Applied to a call option, these **Greeks** are defined as:

$$\Delta_c = \frac{\partial C}{\partial S} \qquad \Gamma_c = \frac{\partial \Delta}{\partial S} \qquad \theta_c = -\frac{\partial C}{\partial t^*}$$

More Greeks for Individual Call Options

- Delta, gamma and theta are the most commonly referenced partial derivatives
- There are a sizable number of other partial derivatives which could also be of value for certain types of situations.
- Vega (for volatility changes); Rho (for interest rates)

$$\frac{\partial C}{\partial \sigma} \qquad \frac{\partial \theta}{\partial t} \qquad \frac{\partial^2 C}{\partial \sigma \, \partial S} \qquad \frac{\partial C}{\partial r}$$

Solving Greeks for Put Options

- Greeks for puts can be solved in terms of Greeks for calls using put-call parity

For the non-dividend case, take the relevant partial derivative in

$$P = C + X\,PV[r, \tau] - S$$

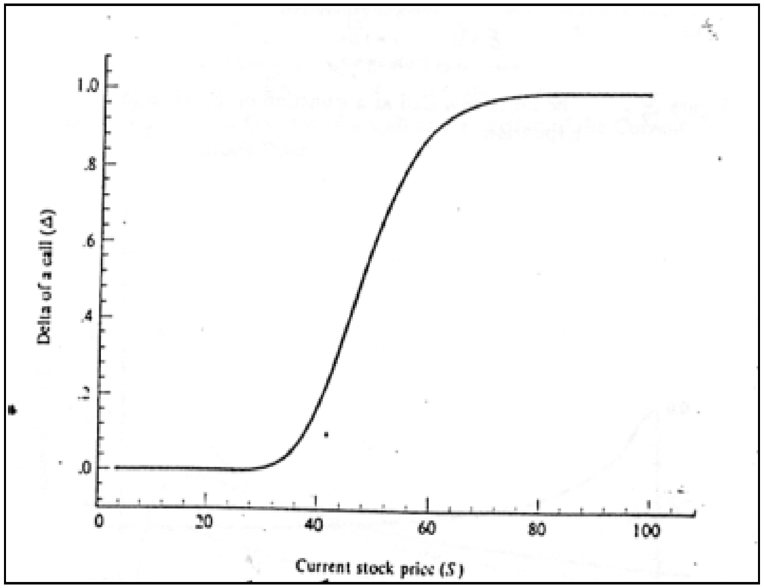
$$\Delta_P = \frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - 1 = \Delta_C - 1$$

$$\Gamma_P = \frac{\partial \Delta_C}{\partial S} = \Gamma_C \qquad \Theta_P = -\frac{\partial P}{\partial t^*} = -\{\frac{\partial C}{\partial t^*} - rX\,e^{-rt^*}\}$$

Specific Greeks: Call Delta

Δ_C, the sensitivity of the call price to changes in the stock price = $N[d_1]$.

$X = 50 \quad t^* = .4 \quad r = .06 \quad \sigma = .3$

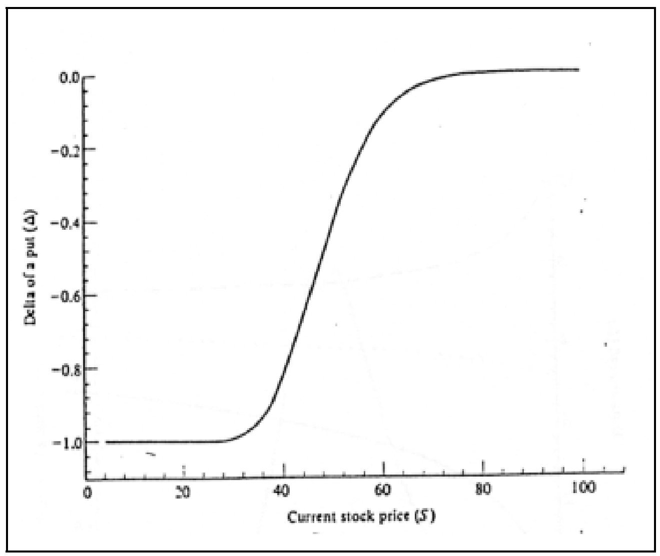


Specific Greeks: Put Delta

Delta for a put, $\Delta_P = N[d_1] - 1$ follows from taking the partial derivative of the put option pricing formula:

$$P = S\,(N[d_1] - 1) - X\,e^{-rt^*}\,(N[d_2] - 1)$$

$$= X\,e^{-rt^*}\,N[-d_2] - S\,N[-d_1]$$



Delta for the Riskless Hedge Portfolio

Consider the number of written ***at-the-money*** options required to form a riskless hedge for different times to expiration:

$r = .06 \quad \sigma = .3 \quad X = S$

Time to Expiration, t*	d_1	$N[d_1]$	$N[d_1]^{-1}$ = # of Options
5 years	.783	.5283	1.893
1 year	.350	.5137	1.947
6 months	.248	.5098	1.962
3 months	.175	.5069	1.973
1 month	.101	.5044	1.983

Specific Greeks: Gamma

- The gamma Γ_c (= Γ_p) of the position measures the sensitivity of the delta to changes in stock prices:

- Observe the derivative of the cumulative normal distribution function is directly related to the normal density function.

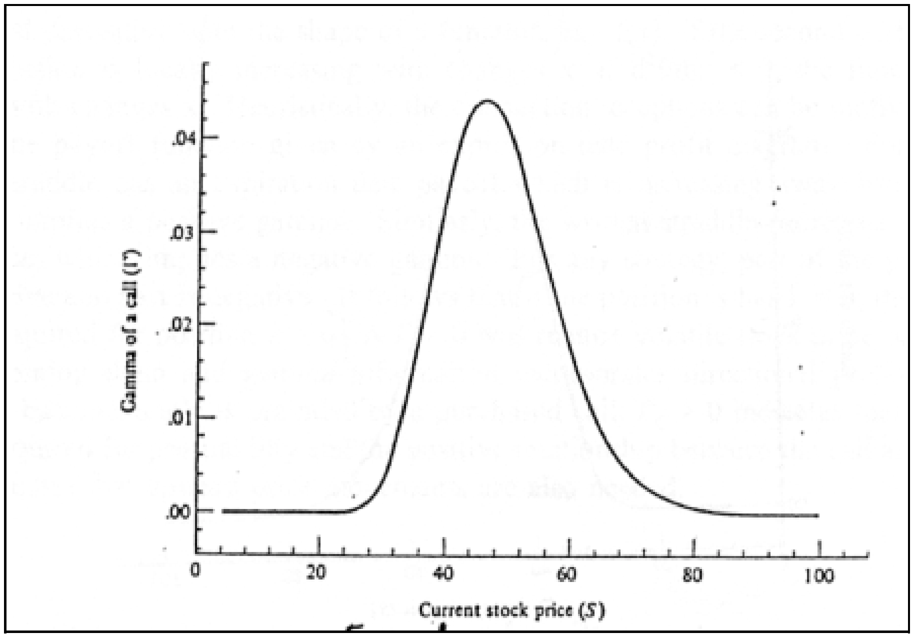
$$\Gamma_c = \frac{\partial \Delta_c}{\partial S} = \frac{\partial^2 C}{\partial S^2} = \frac{1}{S\,\sigma\sqrt{t^*}} N'[d_1] = \frac{1}{S\,\sigma\sqrt{t^*}}\, n[d_1]$$

$$\text{where:} \qquad N'[d_1] = \frac{1}{\sqrt{2\,\pi}} \exp\{\frac{-d_1^2}{2}\} = n[d_1]$$

Diagram for Gamma

- Gamma for puts and calls are related to the normal density function.

X = 50　　*t** = .4　　*r* = .06　　*σ* = .3



Using Gamma Information

- Gamma has two important practical features: size and sign.
- ***For cases such as the riskless hedge portfolio which require rebalancing in order to achieve a delta target***, the size of gamma determines how frequently the position has to be adjusted to maintain the hedge portfolio feature.
- "High" values indicate frequent adjustments are required, "low" values mean the position delta is relatively immune to stock price changes and rebalancing can be done infrequently.
- A ***gamma neutral*** ($\Gamma = 0$) position is one which the delta is 'locally' protected from changes in the stock price.

More about Gamma Information

***The size of gamma for positions which do not have to be rebalanced*.**

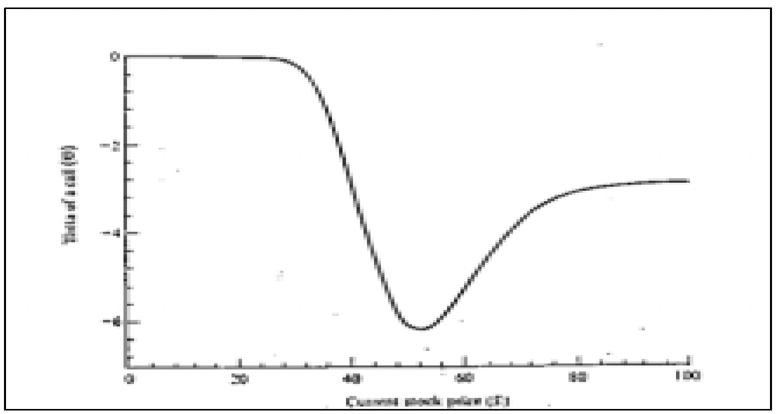
Examples: a portfolio insured with purchased puts or option strategy positions such as straddles and vertical spreads.

These positions are established with a specific delta and gamma provide information about the speed that delta changes as the stock price changes.

Also, ***if the position has $\Gamma < 0$, then stable stock prices are required for position $\pi > 0$***. A $\Gamma > 0$ will require volatile stock price behavior for $\pi > 0$.

Specific Greeks: Call Theta

- The theta (θ_C) of a call measures the sensitivity of the option price to changes in time. Because "time" in options counts backwards, two different definitions of theta are encountered, depending how the impact of time is evaluated.



More on the Call Theta

- The call theta – defined as $(\partial C / \partial t) = \theta_c < 0.$ The analytical solution has both a density and cumulative distribution component.

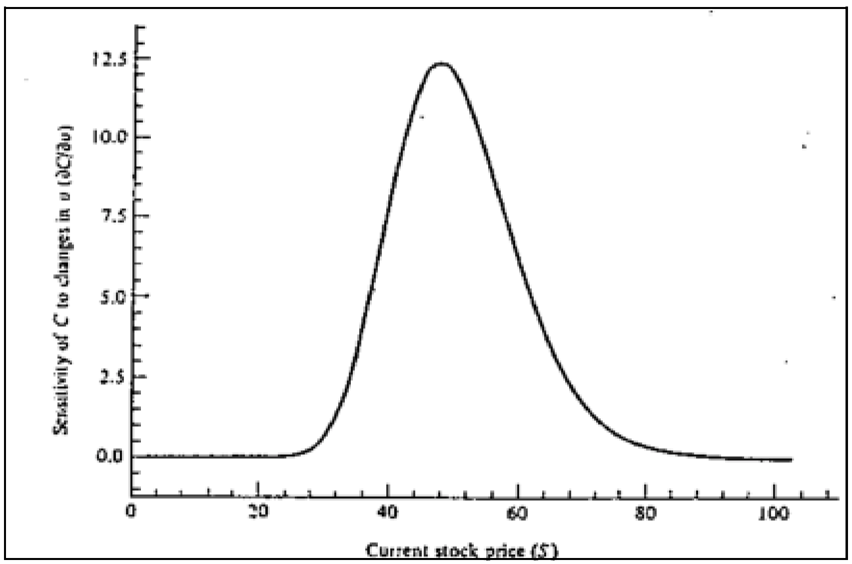
- Theta measures the time decay of the position.

$$\frac{\partial C}{\partial t^*} = \frac{S\sigma}{2\sqrt{t*}} N'[d_1] + X \exp\{-rt^*\} r\, N[d_2] > 0 \quad \rightarrow \quad \theta < 0$$

Specific Greeks: Rho and Vega

- Vega measures the sensitivity to changes in volatility– a density function solution (see diagram).

- Rho measure sensitivity to changes in the riskless rate (an important variable for positions with interest rate options – a distribution function solution.



Verifying Black-Scholes

- Recall the fundamental partial differential equation. (Verify the max boundary condition.)

Given the solutions for delta, gamma and theta – substitute into the PDE to verify the solution.

$$-\frac{\partial C}{\partial t^*} = -\left[\frac{S\sigma}{2\sqrt{t^*}} N'[d_1] + X \exp\{-rt^*\} r\, N[d_2]\right]$$

$$= rSN[d_1] - rX \exp\{-rt^*\} N[d_2]\} - rS\, N[d_1] - \frac{1}{2}\sigma^2 S^2 \frac{1}{S\sigma\sqrt{t^*}} N'[d_1]$$

$$= -rX \exp\{-rt^*\} N[d_2] - \frac{1}{2}\frac{S\sigma}{\sqrt{t^*}} N'[d_1]$$

Greeks for Riskless Hedge Portfolios

- Gamma + Delta Targets
- Setting both the Gamma and Delta of the position both = 0 requires two options – a put and a call.

$$V = S - n_1 C + n_2 P = S - n_1(P - C)$$

$$\Delta_V = 0 = 1 - n_1(\Delta_C - 1 - \Delta_C) \quad \rightarrow \quad n_1 = 1 \quad \rightarrow \quad \Gamma_V = \Gamma_C - \Gamma_C = 0$$

More on Riskless Hedge Portfolios

- Consider hedging the stock position by writing an equal combination of in-the-money and out-of-the-money call options in comparison to using only at the money calls – MUCH BETTER GAMMA.

$$V = S - n_1 C_1 - n_2 C_2 = S - n_1(C_1 + C_2) \qquad \Delta_V = 0 \qquad \rightarrow \qquad n_1 = \frac{1}{\Delta_1 + \Delta_2}$$

$$\Gamma_V = -n_1 (\Gamma_1 + \Gamma_2) = -\frac{\Gamma_1 + \Gamma_2}{\Delta_1 + \Delta_2}$$

Greeks for a Vertical Spread

- Let the vertical spread be long the call at the lower exercise price and short the call at the higher exercise price where:

$$\frac{\partial C_1}{\partial S} = N[d_{1,1}] \qquad \frac{\partial C_2}{\partial S} = N[d_{1,2}]$$

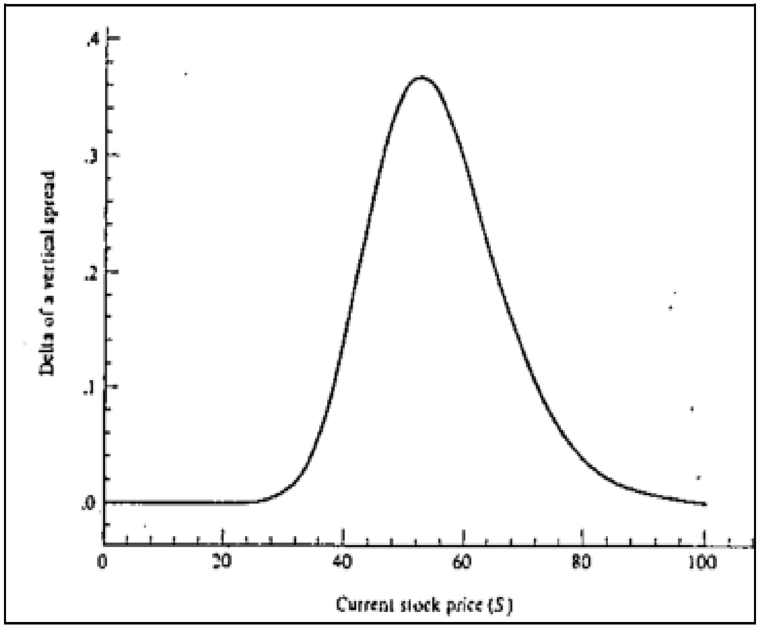
where:

$$d_{1,1} = \frac{\ln\{\frac{S}{X_1}\} + (r + \frac{1}{2}\sigma^2)t^*}{\sigma\sqrt{t*}}$$

$$d_{1,2} = \frac{\ln\{\frac{S}{X_2}\} + (r + \frac{1}{2}\sigma^2)t^*}{\sigma\sqrt{t^*}}$$

Solving for the Delta

- The delta of the vertical spread reduces to the difference of two cumulative distribution functions – this also has the appearance of a density function



Gamma of the Vertical Spread

- Recognizing that gamma is highest for at the money options, the gamma for the vertical spread will be the difference of density functions.

- Exercise: Use delta and gamma information to analyze the straddle, strangle, strap and strip. If the straddle and strangle are both delta neutral, how do the gammas compare?